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Numerical solution of the Orr–Sommerfeld equation using the viscous Green's function and split Gaussian quadrature

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Abstract

We continue our study of the construction of numerical methods for solving two-point boundary value problems using Green's functions, building on the successful use of split Gauss-type quadrature schemes. Here we adapt the method for eigenvalue problems, in particular the Orr–Sommerfeld equation of hydrodynamic stability theory. Use of the Green's function for the viscous part of the problem reduces the fourth-order ordinary differential equation to an integro-differential equation which we then discretize using the split-Gaussian quadrature and product integration approach of our earlier work along with pseudospectral differentiation matrices for the remaining differential operators. As the latter are only second-order the resulting discrete equations are much more stable than those obtained from the original differential equation. This permits us to obtain results for the standard test problem (plane Poiseuille flow at unit stream-wise wave number and Reynolds number 10 000) that we believe are the most accurate to date.

Key words: two-point boundary value problem, eigenvalue problem, Orr–Sommerfeld equation, hydrodynamic stability, Green's function, Gaussian quadrature, product integration
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1 Introduction

The solution of an inhomogeneous linear two-point boundary value problem generally has a Green's integral representation [19, pp. 254–257; e.g.]; that is,

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if $\mathcal{L}$ is a linear ordinary differential operator then the solution ϕ of

$$\mathcal{L}\phi = f \tag{1}$$

satisfying appropriate boundary conditions at $x = a$ and $x = b$ can be written

$$\phi(x) = \int_a^b G(x,\xi) f(\xi)\,\mathrm{d}\xi\,, \qquad (a \le x \le b)\,. \tag{2}$$

If the Green's function for a given operator and boundary conditions is known, this can be used as the basis of a stable and accurate method of computing the solution of the problem [26, 27]. Here we extend this approach to generalized eigenvalue problems of the form

$$\mathcal{L}\phi + \mathcal{K}\phi = c\mathcal{M}\phi\,, \tag{3}$$

where c is the eigenvalue and $\mathcal{K}$ and $\mathcal{M}$ are linear ordinary differential operators of lower order than $\mathcal{L}$ and we assume the Green's function for the related two-point boundary value problem (1) is known. Formal application of (2) leads to

$$\phi(x) + \int_a^b G(x,\xi)(\mathcal{K}\phi)(\xi)\,\mathrm{d}\xi = c\int_a^b G(x,\xi)(\mathcal{M}\phi)(\xi)\,\mathrm{d}\xi\,. \tag{4}$$

The hope is that the integro-differential equation (4) will lead to more stable discrete equations than the original ordinary differential equation (1), as numerical integration is usually more stable than numerical differentiation. In this paper we apply the approach to a famous eigenvalue problem of the form (3), the Orr–Sommerfeld equation of hydrodynamic stability [12, p. 156], and find that this is indeed the case.

1.1 The Orr–Sommerfeld equation

The flow of viscous incompressible fluids is usually described by the Navier–Stokes equations, which comprise a quasilinear second-order vector equation governing the momentum balance and a linear first-order scalar equation governing the conservation of mass [22, p. 2]. Although the system is difficult to solve in general configurations, one-dimensional domains such as the channel between two infinite parallel plates admit simple steady unidirectional solutions such as plane Poiseuille flow [22, p. 4]. Linearizing the momentum equation about such a solution, eliminating the pressure using the conservation of mass, and seeking solutions complex-exponential in time and the stream-wise direction (noting that the system is homogeneous in these coordinates and that the equations have coefficients constant with respect to them) leads to the Orr–Sommerfeld equation [22, p. 7].

The Orr–Sommerfeld equation for a velocity profile $U(x)$, Reynolds number Re, streamwise wavenumber α, and wave speed c is [12, p. 156]

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = \mathrm{i}\alpha\mathrm{Re}\left\{(U-c)\left(\phi'' - \alpha^2\phi\right) - U''\phi\right\} . \tag{5}$$

Here $\phi(x)$ is related to the stream-function perturbation $\delta\psi$ by $\delta\psi(x,y,t) = \phi(x)\exp\{\mathrm{i}\alpha(x-ct)\}$. The boundary conditions for a channel with solid walls at $x = \pm 1$ are

$$\phi(\pm 1) = 0 \qquad (\text{'impermeability'}) \tag{6a}$$
$$\phi'(\pm 1) = 0 \qquad (\text{'no-slip'}) . \tag{6b}$$

The Reynolds number Re is defined as the centre-line velocity times the channel width divided by the kinematic viscosity (in any consistent set of physical units); thus the left-hand side represents the effects of viscosity. Without viscosity, $\mathrm{Re}^{-1} = 0$, the left-hand side is dropped, and only the part of (5) in braces remains: the Rayleigh equation [12, p. 130]. The Rayleigh equation is only second order, so the no-slip boundary conditions (6b) have to be dropped, and further the coefficient of the second derivative vanishes wherever $U(x) = c$; these singularities do not occur in the Orr–Sommerfeld equation, being smeared out by viscosity into boundary layers and critical layers, respectively. The Rayleigh and Orr–Sommerfeld equations have been the subject of much study in hydrodynamics [22, 7, 12].

Both the Orr–Sommerfeld and Rayleigh equations are homogeneous in ϕ, and are therefore usually solved as eigenvalue problems for unknown complex c given real α (the 'temporal' problem) or complex α given real frequency $-\alpha c$ (the 'spatial' problem) [12, p. 152]. Here we consider only the temporal problem, for which the left-hand side contains a fourth-order ordinary differential operator with known real constant coefficients.

For many velocity profiles of interest, such as plane Poiseuille flow for which $U(x) = 1 - x^2$, the physically relevant Reynolds numbers are quite large; for example, eigenvalues c with positive imaginary part, denoting exponential growth of disturbances in time, only occur for real wave numbers α if $\mathrm{Re} > 5772.2$ [12, p. 192]. This means that the boundary and critical layers are thin which implies difficulty in numerically solving (5). Many different approaches have been tried, including compact finite difference [33], Galerkin [11], shooting [30], Chebyshev-τ [31], stabilized shooting [8], and pseudospectral [18, 34, 25, 24]. These diverse methods have achieved varying degrees of accuracy and efficiency.

To date, to our knowledge, the most accurate results have been obtained by the various 'spectral' methods [31, 18]; however, we have found empirically that they are not numerically stable, that is, the error ultimately grows with the

number of nodes or expansion terms. For many problems, this transition occurs at such a high number and the convergence is initially so rapid that it does not prevent very accurate results being obtained; nevertheless, numerically stable methods are more robust and reliable in practice, and so the search for one is not without value. In earlier studies [26, 27] we found the Green's function integral expression for the solution of a two-point boundary value problem to be very stable, and we developed quadrature techniques to render it accurate and rapidly convergent too. In this paper we adapt this approach to eigenvalue problems and apply it to the Orr–Sommerfeld equation for a standard test case: plane Poiseuille flow at $\mathrm{Re} = 10^4$ and $\alpha = 1$ [33, 14, 31, 36, 4, 18, 34].

2 Recasting the Orr–Sommerfeld equation

2.1 The viscous Green's function

As discussed by Li [21], the solution of the two-point boundary value problem

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = f\,, \tag{7}$$

in the channel with boundary conditions (6) can be expressed in terms of its Green's function G as

$$\phi(x) = \int_{-1}^{1} G(x,\xi) f(\xi)\,\mathrm{d}\xi\,. \tag{8}$$

In general, Green's functions for fourth-order ordinary differential equations can be written in the piecewise-degenerate form

$$G(x,\xi) = \begin{cases} +u_{1L}(x)v_{1L}(\xi) + u_{2L}(x)v_{2L}(\xi)\,, & -1 < \xi < x \\ -u_{1R}(x)v_{1R}(\xi) - u_{2R}(x)v_{2R}(\xi)\,, & x < \xi < 1\,, \end{cases} \tag{9}$$

where the u are linearly independent solutions of (7) with $f = 0$, the u_L and u_R satisfy the boundary conditions at $x = +1$ and $x = -1$ respectively, and the v are chosen so that $G(x,\xi)$ is smooth up to a jump discontinuity in its third derivative with respect to x as ξ crosses $\xi = x$. Different but equivalent expressions for G follow from different choices of the u; two such were derived by Li [21] and McBain & Armfield [23]. Another results from

$$u_{1L}(x) = \alpha(1-x)\sinh\alpha(1-x) \tag{10a}$$
$$u_{2L}(x) = \alpha(1-x)\cosh\alpha(1-x) - \sinh\alpha(1-x) \tag{10b}$$
$$u_{1R}(x) = \alpha(1+x)\sinh\alpha(1+x) \tag{10c}$$
$$u_{2R}(x) = \alpha(1+x)\cosh\alpha(1+x) - \sinh\alpha(1+x) \tag{10d}$$

for which the remaining factors are given by

$$\begin{aligned} Wv_{1L}(\xi) =& \alpha^4(1-\xi)\left\{\cosh\alpha(1-\xi) - \cosh\alpha(3+\xi)\right\} && \text{(11a)}\\ &+ 4\alpha^5(1+\xi)\sinh\alpha(1-\xi) \\ Wv_{2L}(\xi) =& \alpha^4\left\{(1-\xi)\sinh\alpha(3+\xi) - (3+\xi)\sinh\alpha(1-\xi)\right\} && \text{(11b)}\\ &- \alpha^3\left[\left\{1+4\alpha^2(1+\xi)\right\}\cosh\alpha(1-\xi) - \cosh\alpha(3+\xi)\right] \\ Wv_{1R}(\xi) =& \alpha^4(1+\xi)\left\{\cosh\alpha(3-\xi) - \cosh\alpha(1+\xi)\right\} && \text{(11c)}\\ &- 4\alpha^5(1-\xi)\sinh\alpha(1+\xi) \\ Wv_{2R}(\xi) =& \alpha^4\left\{(3-\xi)\sinh\alpha(1+\xi) - (1+\xi)\sinh\alpha(3-\xi)\right\} && \text{(11d)}\\ &+ \alpha^3\left[\left\{1+4\alpha^2(1-\xi)\right\}\cosh\alpha(1+\xi) - \cosh\alpha(3-\xi)\right] \end{aligned}$$

where W is the Wronskian of the complementary functions (10)

$$W \equiv W(u_{1L}, u_{2L}, u_{1R}, u_{2R}) = -2\alpha^6(\cosh 4\alpha - 8\alpha^2 - 1)\,. \qquad (12)$$

2.2 An integro-differential Orr–Sommerfeld equation

With the Green's function defined by (8), the Orr–Sommerfeld equation (5) with boundary conditions (6) is converted to integro-differential form [21]:

$$\begin{aligned} \int_{-1}^{1} G(x,\xi)\left[U(\xi)\left\{\alpha^2\phi(\xi) - \phi''(\xi)\right\} + U''(\xi)\phi(\xi)\right]\mathrm{d}\xi - \frac{\mathrm{i}\phi(x)}{\alpha\mathrm{Re}} \\ = c\int_{-1}^{1} G(x,\xi)\left\{\alpha^2\phi(\xi) - \phi''(\xi)\right\}\mathrm{d}\xi\,. \qquad (13) \end{aligned}$$

Li [21] went further and integrated by parts to eliminate the remaining differential operators. Another way to obtain a pure integral equation would be to replace the dependent variable with its preimage under the Green's function; that is, replace $\phi(x)$ with $\int_{-1}^{1} G(x,\xi)\sigma(\xi)\,\mathrm{d}\xi$ and solve for σ [32, 15].

Here however we do not pursue these approaches and rather discretize (13), using pseudospectral differentiation matrices [34] for the differential operators. The advantage of discretizing (13) rather than the original differential Orr–Sommerfeld equation (5) is that only second-order derivatives appear in (13) whereas (5) is fourth-order and it is well known that the condition of pseudospectral differentiation matrices deteriorates rapidly with increasing order of differentiation [3]. Indeed the results reported in § 4 demonstrate that the discretization of (13) is more stable than that of (5).

3 Discretization

3.1 Formal discretization

Given a set of n collocation points $\{x_i\}_{i=1}^n$, we intend to use matrices to replace Green's function integrals

$$\int_{-1}^{1} G(x_i,\xi) f(\xi)\,\mathrm{d}\xi \doteq \sum_{j=1}^{n} g_{ij} f_j \tag{14}$$

and the second derivative

$$f''(x_i) \doteq \sum_{j=1}^{n} D^{(2)}_{ij} f_j\,. \tag{15}$$

Here f_j stands for the ordinate $f(x_j)$ and the coefficients g_{ij} and $D^{(2)}_{ij}$ are yet to be specified. This will reduce the integro-differential Orr–Sommerfeld equation (13) at the collocation points to an algebraic generalized eigenvalue problem

$$\sum_{j=1}^{n} l_{ij}\phi_j = c\sum_{j=1}^{n} m_{ij}\phi_j \qquad (i=1,2,\ldots,n) \tag{16}$$

with coefficients

$$l_{ij} = \sum_{k=1}^{n} g_{ik}\left[U_k\left\{\alpha^2\delta_{kj} - D^{(2)}_{kj}\right\} + U''_k\delta_{kj}\right] - \frac{\mathrm{i}\delta_{ij}}{\alpha\mathrm{Re}} \tag{17a}$$

$$m_{ij} = \sum_{k=1}^{n} g_{ik}\left\{\alpha^2\delta_{kj} - D^{(2)}_{kj}\right\}, \tag{17b}$$

where δ_{ij} is Kronecker's delta.

3.2 Split product integration

If $\{\lambda_j\}_{j=1}^n$ is a set of sampling functions for the collocation points, that is a set with the sampling property

$$\lambda_j(x_i) = \delta_{ij}\,, \tag{18}$$

then an arbitrary function $f(x)$ has the interpolant $\sum_{j=1}^n \lambda_j(x) f_j$. We can thus determine the g_{ij} to give an interpolatory-type integration formula [10, p. 27] by making the formula (14) correct when f is replaced by each of the sampling functions λ_j:

$$g_{ij} \equiv \int_{-1}^{1} G(x_i,\xi)\lambda_j(\xi)\,\mathrm{d}\xi\,. \tag{19}$$

Since the Green's function is smooth off $\xi = x$, high-order quadrature schemes can be applied after splitting:

$$g_{ij} = \int_{-1}^{x_i} G(x_i,\xi)\lambda_j(\xi)\,\mathrm{d}\xi \qquad + \int_{x_i}^{1} G(x_i,\xi)\lambda_j(\xi)\,\mathrm{d}\xi \tag{20a}$$

$$\doteq \sum_k w_{ik}^L G(x_i, x_{ik}^L)\lambda_j(x_{ik}^L) \qquad + \sum_\ell w_{i\ell}^R G(x_i, x_{i\ell}^R)\lambda_j(x_{i\ell}^R)\,. \tag{20b}$$

The abscissae x_{ik}^L and $x_{i\ell}^R$ and weights w_{ik}^L and $w_{i\ell}^R$ will be discussed in § 3.5.1.

3.3 Differentiation matrices

The interpolatory approach used to derive (19) can also be used for derivatives; in which case, for example, for (15), $D_{ij}^{(2)} = \lambda_j''(x_i)$. This approach has a shortcoming, however: the differentiation matrices are usually singular by design. For example, if the sampling functions are chosen to span the space of polynomials of degree less than n, the kth matrix eliminates the vectors of ordinates of polynomials of degree less than k and so has rank $n-k$. In particular, the second differentiation matrix has rank $n-2$, which remains after α times the identity matrix is subtracted from it. The result is that the mass matrix (17b) will be singular which will lead to a very poor condition of the eigenvalues of (16).

The remedy is to impose boundary conditions on the differentiation matrix. This can be done by coercing the sampling functions to satisfy the boundary conditions; thus

$$\hat\lambda_j(x) \equiv \frac{\omega(x)}{\omega(x_j)}\lambda_j(x) \tag{21}$$

retains the sampling property (18). For example, if

$$\omega(x) = 1 - x^2 \tag{22}$$

then $\hat\lambda_j(\pm1) = 0$ so that any interpolant $\sum_{j=1}^n \hat\lambda_j(x) f_j$ satisfies (6a). Similarly if $\omega(x) = (1-x^2)^2$ then any interpolant satisfies (6).

Although the eigenfunction ϕ in (5) or (13) satisfies (6), these should not both be imposed on the second-order differential operator, or spurious eigenvalues will intrude; instead, only (6a) should be enforced, using (22) [18, 34]. This can be made plausible either physically, considering that the operator really belongs to the inviscid Rayleigh equation for which the no-slip condition is inapplicable, or in terms of matrix rank, since imposing four boundary conditions on a second-order differentiation matrix must lead to rank deficiency and therefore a singular mass matrix (17b).

3.4 The particular discretization

To complete the specification of the discretization, we need to choose the collocation points, the secondary quadrature rules, and the sampling functions. Here we choose the $\{x_i\}_{i=1}^n$ as the abscissae of the n-point Gaussian quadrature rule, choose this same rule for the $2n$ secondary integrations (20), and we choose the Lagrangian sampling polynomials—the unique set of polynomials of degree $n-1$ with the sampling property (18).

3.5 Computing the discretization

To assemble the coefficients (17), we need to compute the secondary quadrature abscissae and their weights, evaluate the Green's function and sampling polynomials there, and evaluate the second derivatives of the sampling functions at the collocation points.

3.5.1 Collocation points and quadrature rules

Although they are tabulated [9, pp. 916–919], here the Gaussian quadrature weights and abscissae were conveniently computed from the eigendecomposition of the Jacobi matrix [20].

Since the collocation points $\{x_i\}_{i=1}^n$ are abscissae of a quadrature rule on $-1 < x < 1$ and if their corresponding weights are $\{w_i\}_{i=1}^n$ so that

$$\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi \doteq \sum_{i=1}^{n} w_i f_i\,, \tag{23}$$

the secondary quadrature rules can be obtained by affine mapping for $k = 1, 2, \ldots, n$,

$$x_{ik}^L = \frac{1}{2}(1+x_i)(1+x_k) - 1 \qquad w_{ik}^L = \frac{1}{2}(1+x_i)w_k \tag{24}$$

$$x_{ik}^R = 1 - \frac{1}{2}(1-x_i)(1-x_k) \qquad w_{ik}^R = \frac{1}{2}(1-x_i)w_k\,. \tag{25}$$

This is just as in our earlier work [27].

3.5.2 Polynomial interpolation and differentiation

The sampling polynomials are evaluated at the secondary quadrature abscissae using their 'barycentric representation'

$$\lambda_j(x) = \frac{b_j}{x - x_j} \div \sum_{k=1}^{n} \frac{b_k}{x - x_k} \tag{26}$$

where

$$b_j \equiv \prod_{\substack{k=1 \\ k \neq j}}^{n} (x_k - x_j)^{-1} \,. \tag{27}$$

as its stability and efficiency are well known [17, 2].

For the Lagrangian sampling polynomials, the off-diagonal elements of the differentiation matrices can be obtained from [35]

$$\lambda_j^{(k)}(x_i) = \frac{k}{x_i - x_j} \left\{ \frac{b_j}{b_i} \, \lambda_i^{(k-1)}(x_i) - \lambda_j^{(k-1)}(x_i) \right\} \qquad (j \neq i) \tag{28}$$

and the diagonal elements from [18]

$$\lambda_i^{(k)}(x_i) = \sum_{m=1}^{k} (-1)^{m-1} \frac{(k-1)!}{(k-m)!} \lambda_i^{(k-m)}(x_i) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(x_i - x_j)^m} \,. \tag{29}$$

These recurrence relations in the order of differentiation k can begin with $k = 0$ for which the differentiation matrix is just the identity matrix (18).

Once we have computed the differentiation matrices corresponding to a set of sampling functions, the modified differentiation matrices for coerced sampling functions (30) can be computed from them using Leibniz's generalized product rule

$$\hat{\lambda}_j^{(k)}(x_i) = \frac{\omega(x_i)}{\omega(x_j)} \sum_{m=0}^{k} \binom{k}{m} \frac{\omega^{(k-m)}(x_i)}{\omega(x_i)} \lambda_j^{(m)}(x_i) \,, \tag{30}$$

although direct techniques replacing (28)–(29) also exist [35, 34].

3.6 Implementation

The implementation of the above method in GNU Octave [13] is reasonably straightforward, except that for large values of n a fully vectorized version of the code would require a large amount of random access memory: note that there are $2n^3$ different summands in (20). To relieve this, at some expense in speed, the Green's function integral coefficients were computed one column at a time, iterating over j in (20).

Following Gary & Helgason [14], we convert the algebraic generalized eigenvalue problem (16) to the standard form $(M^{-1}L)\phi = c\phi$, which is possible provided the mass matrix with coefficients (17b) is nonsingular (§ 3.3). We note that there do exist algorithms, such as QZ [28], for solving the generalized problem without this conversion.

The standard-form algebraic eigenvalue problem was then solved with GNU Octave's `eig` function [13], which uses the QR Schur factorization, as implemented in LAPACK's `ZGEEV` subroutine [1].

4 Results

We present the results of the method for the standard benchmark problem for the Orr–Sommerfeld equation: the eigenvalue of largest imaginary part (the 'leading' eigenvalue, hereafter) for plane Poiseuille flow at $\mathrm{Re} = 10^4$ and $\alpha = 1$. As n increases from 30 to 60, the method converges rapidly to within 10^{-12} of a limiting value. The convergence is oscillatory. Further increase in n does not lead to any increase in accuracy, but rather a slowly increasing noisy error contaminates the result. The method is thus not strictly stable, though is much less unstable than its alternatives, as will be shown below.

The estimate

$$c \doteq 0.237526488821 + 0.003739670623\mathrm{i}\,, \tag{31}$$

obtained by Chubb [6] using the present method and fitting a line to the data for several n near $n = 60$, the narrowest part of the envelope. It is believed to be accurate to the number of stated decimal places, and is believed to be the most accurate estimate to date, adding an extra figure to the 1994 estimate by Huang & Sloan [18]. A chronology of increasingly accurate estimates is given in table 1, extending that compiled by Brenier et al. [4] in 1986.

To investigate the convergence behaviour of the present method, we plot in figure 1 the absolute error in the imaginary part of the leading eigenvalue, estimated as the discrepancy with (31). As mentioned, at least 20 collocation points are required to begin to resolve the eigenfunction, rapid convergence is obtained as n increases from 30 to 60, and for higher n the accuracy slowly deteriorates. Because the convergence is oscillatory, for a few particular values of n the discrepancy of the value from (31) is less than the claimed accuracy but this does not justify a claim for greater accuracy.

Figure 1 also contains results for a number of alternative methods, using results published by Thomas [33] and Orszag [31] and a demonstration program accompanying the MATLAB Differentiation Suite [34], as well as two in-house programs.

INVESTIGATORS	YEAR	$\Re c$	$\Im c$
Thomas [33]	1953	0.2375*259*	0.0037404
Gary & Helgason [14]	1970	0.2375265*0*	0.0037396*9*
Orszag [31]	1971	0.23752649	0.00373967
Brenier et al. [4]	1986	0.23752649	0.0037396729
Huang & Sloan [18]	1994	0.23752648882	0.00373967062
Chubb [6]	2006	0.237526488821	0.003739670623

Table 1
Historical comparison of results for the Orr–Sommerfeld equation for Poiseuille flow at $\alpha = 1$ and $\mathrm{Re} = 10^4$, extending table 7 of Brenier et al. [4]; italics denote dubious digits.

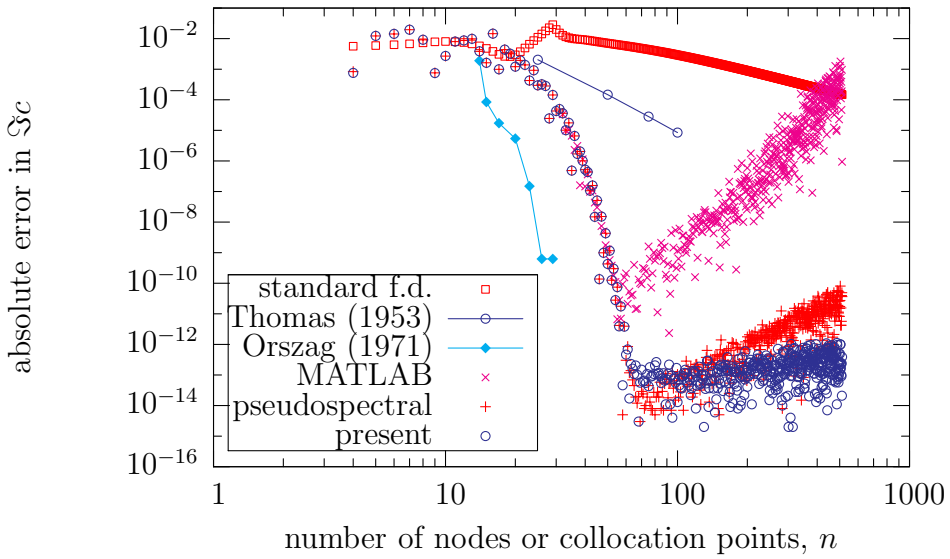

Fig. 1. Convergence for the Orr–Sommerfeld benchmark problem of the present method and a number of alternatives as discussed in § 4.

We notice that all the methods except Orszag's [31] Chebyshev-τ require about 20 nodes for resolution before 'asymptotic' convergence is entered. Orszag [31], exploited the even symmetry $\phi(-x) = \phi(x)$ of the leading eigenfunction; therefore his results have a natural factor of two advantage in n . This symmetry could easily be exploited in the present method, but wasn't as it does not apply in other Orr–Sommerfeld problems, such as boundary layers [29, 24].

The standard (in-house) and compact finite difference schemes [33] attain their expected second- and fourth-order accuracy. The other methods, all spectral, converge at about the same rate, allowing for the factor of two advantage of Orszag [31]. The levelling-off of Orszag's results is due simply to his reporting only eight decimal places.

The program `orrsom` accompanying the MATLAB Differentiation Matrix Suite [34] suffers rather badly from the effects of round-off as n increases past 60 to the extent that for $n > 500$ it is less accurate than the second-order finite difference scheme. Investigation has shown that this is partly due to MATLAB's `eig` function for the algebraic generalized eigenvalue problem, and so is not intrinsic to the pseudospectral differentiation matrix approach. Converting the generalized eigenvalue problem to standard form as described in § 3.6 improves the results significantly, bringing them more into line with the in-house Octave results marked 'pseudospectral' in figure 1. Those results were obtained using differentiation matrices computed as described in § 3.5.2 and based on the Gauss quadrature points (rather than the Chebyshev points of the MATLAB program [34], though this does not make a significant difference).

The present method is much more robust with respect to round-off, as may be seen by the flatness of its error in figure 1 for $n > 10^2$ compared to the two pseudospectral schemes. In particular, we envisage that if another problem was considered that required considerably more than twenty nodes for resolution, say more than 10^2, then the pseudospectral schemes could have difficulty obtaining accurate results. Moreover, due to the strong erratic variation of their results with n for large n, it may be difficult to extract the result and an estimate of its accuracy, requiring runs at several values of n. This significantly increases the computational cost.

We expect that the ultimate accuracy of determination of the eigenvalue is limited by the underlying condition of the continuous problem, and that any discretization will have a condition that is only worse (as discussed by Hackbusch [16, p. 65] for discretizations of integral equations). It is not that the pseudospectral methods cannot obtain individual results as accurate as the Green's function method, it's just that if they have enough n to have converged to that level of accuracy they will already have accumulated so much random round-off error that the spread in results for nearby n is too large to permit a confidence in any one result at a particular n.

The same stability advantages are expected for other eigenvalue problems. In assessing the competitiveness of the method though, account must be taken of the relatively expensive computation of the coefficients (20). Moreover, unlike other methods, this one relies on a relevant Green's function being available; although conceptually straightforward, constructing these for higher-order two-point boundary value problems is arduous.

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*** Revision Notes**

I accept the reviewer's two points and have modified the paper accordingly. I have also made two very minor deletions.

*** Revision Notes**

* p. 4: abscissae -> nodes, following reviewer's suggestion

* p. 4, eq. 10d: deleted spurious trailing full stop

* p. 11: deleted `and its alternatives' in first paragraphs as this is dealt with later on

* p. 11: abscissae -> collocation points, following reviewer's suggestion

* pp. 10, 11: added discussion of how Chubb obtained his estimate and why though some values for particular n are closer to that than the claimed accuracy of the method that does justify a claim for greater accuracy. Following reviewer's suggestion.

* pp. 11, 12: abscissae -> nodes, twice following reviewer's suggestion