The Conduction and Convection Regimes in a Cavity with Evenly Heated and Cooled Vertical Walls

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Abstract

We treat natural convection in a plane vertical rectangular cavity with one vertical wall evenly heated, the other cooled, and an adiabatic floor and ceiling. We predict the temperature difference across the cavity by simple analysis, and present numerical solutions verifying the results.

If the cavity is infinitely tall, there is an exact steady one-dimensional solution depending on a single parameter: the stratification. This quantity is not prescribed by the boundary conditions, but can be determined analytically from an energy balance on a suitable control volume. This allows the calculation of the cross-cavity temperature difference (the Nusselt number–Rayleigh number relation) over the entire laminar range of Rayleigh numbers, for all Prandtl numbers, and for all sufficiently large aspect ratios. The result links the trivial conduction limit to the Kimura–Bejan boundary layer approximation, which is also shown to be asymptotically correct.

Numerical solutions verify the conduction–convection transition for the Nusselt number. They also indicate the dependence on the parameters of the problem (Rayleigh and Prandtl numbers) of the bound on the aspect ratio for the applicability of the approach: that is, the minimum separation of ceiling from floor which permits a fully developed region.

Introduction

Natural convection in a side-heated vertical rectangular cavity is one of the classical problems of heat transfer. Most previous studies [2, 3, 4] have treated the case in which the vertical walls are held at different fixed uniform temperatures; however, a slight variation perhaps more closely related to some real-world configurations and possessing several interesting theoretical properties is that first investigated by Kimura & Bejan [1]: a uniform heat flux is specified on the vertical walls, positive into the cavity on one side and equal and positive out of the cavity on the other; see Fig. 1.

In particular, a uniform heat flux is a better model than a uniform temperature of solar or electrical resistance heating; a negative uniform heat flux is a good model for radiative cooling to deep space. The mass transfer analogue also arises in certain electrochemical processes [5].

The interesting properties of the configuration discovered by Kimura & Bejan [1] (*KB* hereafter) are that, in the laminar convection regime, the vertical wall boundary layer thicknesses are uniform, that the core is stagnant and linearly stratified, and that the temperature on the vertical walls increases with height at the same rate as it does in the stably stratified core. This is to be contrasted with the laminar convection regime in the side-heated cavity with isothermal walls. There the vertical wall boundary layers vary in thickness along their length, first drawing then disgorging fluid. While relatively calm, the core is still in motion as a result of this entrainment and discharge, and although generally stably stratified, the core isotherms are tilted and the vertical core temperature gradient is a complicated function of the flow and heat transfer parameters. And of course the vertical wall temperatures don't vary like that in the core: they're uniform.

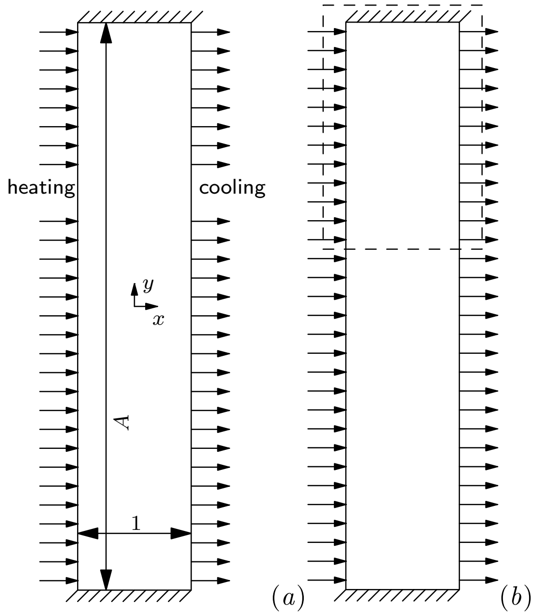

Fig. 1 Configuration (a); the energy control volume [1, Fig. 2] (b).

Another especially interesting property is that a simple closed form solution can be found, valid far from both the floor and ceiling. This goes beyond KB in that the present solution applies for all Rayleigh and Prandtl numbers (provided the resulting laminar solution is stable); further, the present solution is exact, except in its failure to satisfy the boundary conditions at the floor and ceiling. In this paper, we derive the one-dimensional solution, and investigate how well it describes steady laminar two-dimensional natural convection, as known from specially obtained numerical solutions.

Formulation

If lengths are scaled with the cavity width, velocities with the fluid thermal diffusivity over the cavity width, and temperature differences with the product of the cavity width and the imposed normal temperature gradient, the dimensionless governing Oberbeck–Boussinesq equations are

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\frac{1}{\mathrm{Pr}}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \mathrm{Ra}\, T\mathbf{j} + \boldsymbol{\nabla}^2\boldsymbol{u} \tag{2}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \nabla^2 T. \tag{3}$$

The boundary conditions are that $\boldsymbol{u} = \mathbf{0}$ on all four walls while

$$\partial T/\partial x = -1 \qquad (x = \pm 1/2) \tag{4}$$

$$\partial T/\partial y = 0 \qquad (y = \pm A/2). \tag{5}$$

Since the purely Neumann boundary conditions (4)–(5) only determine the temperature to within an additive constant, we also require, for example, $\iint T \,\mathrm{d}x\,\mathrm{d}y = 0$.

One-dimensional solution

Equations (1)–(3) with (4) but not (5) admit a family of one-dimensional solutions depending on a stratification parameter s [6]:

$$\boldsymbol{u} = \frac{\sinh s(1-2x)\sin s(1+2x) - \sinh s(1+2x)\sin s(1-2x)}{16s^3(\sinh 2s + \sin 2s)/\mathrm{Ra}}\,\mathbf{j} \tag{6}$$

$$T = \frac{\cosh s(1-2x)\cos s(1+2x) - \cosh s(1+2x)\cos s(1-2x)}{2s(\sinh 2s + \sin 2s)} + \frac{64s^4}{\mathrm{Ra}}\,y. \tag{7}$$

We expect this solution, which is also a special case of one known to Ostroumov [7, p. 58], to exist over some range of y between $\pm A/2$, provided A is large enough (so that there is a region far from both the floor and ceiling) and Ra small enough (so that the laminar solution is stable).

Apparently unaware of the work of Lietzke [6], KB simplified the governing equations in three steps: first, the usual boundary layer approximation; second, assuming $\mathrm{Pr} \to \infty$ and dropping the inertial term in (2); and third, the 'modified Oseen method' [8], i.e. the replacement of nonlinear terms and those with variable coefficients with constant-coefficient linear terms. They arrived at a solution equivalent to

$$v \approx \frac{\mathrm{e}^{-s(1+2x)}\sin s(1+2x) - \mathrm{e}^{-s(1-2x)}\sin s(1-2x)}{16s^3/\mathrm{Ra}} \tag{8}$$

$$T \approx \frac{\mathrm{e}^{-s(1+2x)}\cos s(1+2x) - \mathrm{e}^{-s(1-2x)}\cos s(1-2x)}{2s} + \frac{64s^4}{\mathrm{Ra}}\,y. \tag{9}$$

This can be more directly obtained as the $s \to \infty$ asymptotes of (6) and (7) [9].

Nevertheless, KB's next step is crucial: integrate the heat equation (3) over a rectangular control volume enclosing either the floor or (as shown in Fig. 1 b) the ceiling of the cavity. Since the energy flux through the cavity boundary is everywhere prescribed, the integral reduces to the requirement that the net energy flux through any horizontal surface cutting the cavity must vanish. If we assume Lietzke's one-dimensional solution applies there, we find

$$\begin{aligned}\mathrm{Ra}^2 = &2^{14}s^9(\sinh 2s + \sin 2s)^2 \div \\ &\{(\sinh 2s + \sin 2s)(\cosh 2s - \cos 2s) \\ &-4s\sinh 2s \sin 2s\}\,.\end{aligned} \tag{10}$$

Any non-negative stratification parameter s corresponds to some non-negative Rayleigh number and so the core solution is determined for all Pr, Ra, and A, without recourse to details of the solutions in the floor and ceiling regions, provided only that the one-dimensional solution exists at some range of heights.

For large s, the energy balance reduces to [1]

$$s \sim \mathrm{Ra}^{2/9}/2^{14/9}; \tag{11}$$

and using this, the KB solution (8)–(9) has been experimentally verified in the mass transfer analogue [5].

Two-dimensional numerical solutions

Numerical solutions for the full nonlinear two-dimensional problem were obtained using two codes: one based on relaxation of the steady stream-function–vorticity equations, derived from Naylor's ENCLREC [4, pp. 385–403], and the other a transient finite-volume velocity–pressure scheme [10]. Although the former was advertised as 'very simplistic' [4, p. 398], the present computations are not demanding and it proved adequate. A sample solution is shown in Fig. 2.

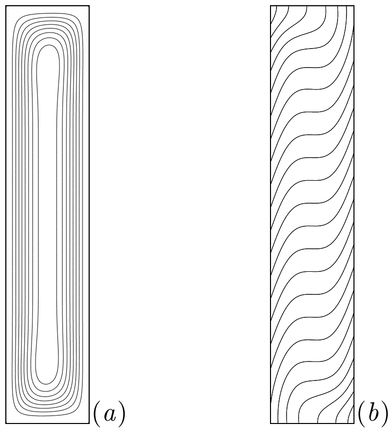

Fig. 2 Stream-lines (a) and isotherms (b) for natural convection in an evenly heated and cooled vertical cavity at $\mathrm{Ra} = 6000$, $\mathrm{Pr} \to \infty$, and $A = 5$.

It can be seen that over much of the height of the cavity, the stream-lines are parallel and vertical and surround a stagnant stratified core; the isotherms have similar shapes and appear evenly spaced in y.

All this suggests the existence, for this set of parameters, of a region in which the one-dimensional solution derived above should exist. We expect that such existence is favoured by larger A, but also anticipate that the least required A may depend on Ra and Pr. In the following two sections we investigate the interaction of A and Ra by examining the difference of the two-dimensional numerical temperature fields from the one-dimensional analytic solution (much as done previously for the isothermal hot-and-cold wall problem [11]).

Tall cavity regime

Figure 3 demonstrates the 'tall cavity regime'.

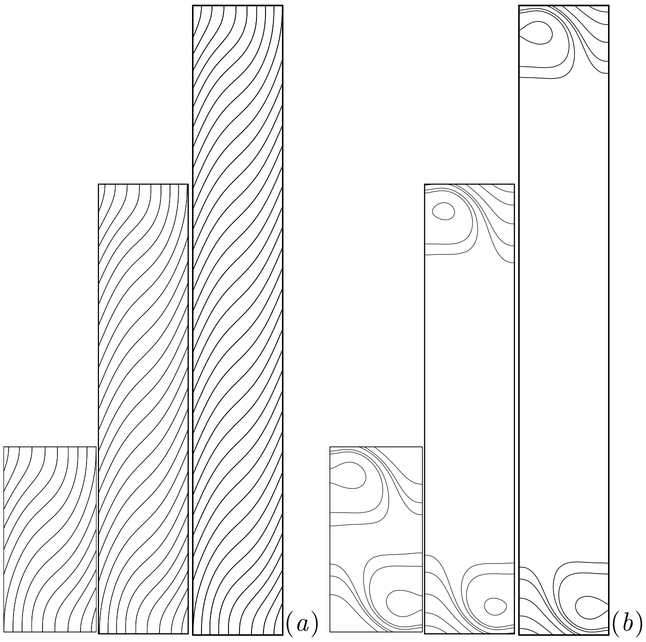

Fig. 3 Increasing A at fixed Ra = 1000 and Pr = ∞; temperatures, isotherms each $\Delta T = 0.1$ (a) and discrepancies of temperature from one-dimensional solution, $\pm 1, 2, 5, 10, 15 \times 10^{-2}$ error contours (b).

For fixed Ra and Pr, increasing A past a certain level simply extends the section over which the one-dimensional solution applies; notice that the floor regions are almost identical. This is an extremely useful result, since it means that from any such solution, the solution for any other larger A can be constructed, merely by inserting a longer one-dimensional region in the middle. This is just as in the isothermal hot-and-cold wall problem [11, 12].

Increasing Rayleigh number

The situation for increasing Rayleigh number at fixed A is different, because in the present problem the one-dimensional solution depends on the Rayleigh number, whereas for the isothermal case a one-dimensional solution is only possible with zero stratification: it is essentially the same as the $s \to 0$ asymptote of (6)–(7):

$$\boldsymbol{u} = \frac{\mathrm{Ra}}{24}\,(4x^3 - x)\,\mathbf{j} \qquad (12)$$

$$T = -x, \qquad (13)$$

with an appropriate redefinition of Ra [2, 3, 12]. The effect of increasing Ra in the present problem is illustrated in Fig. 4.

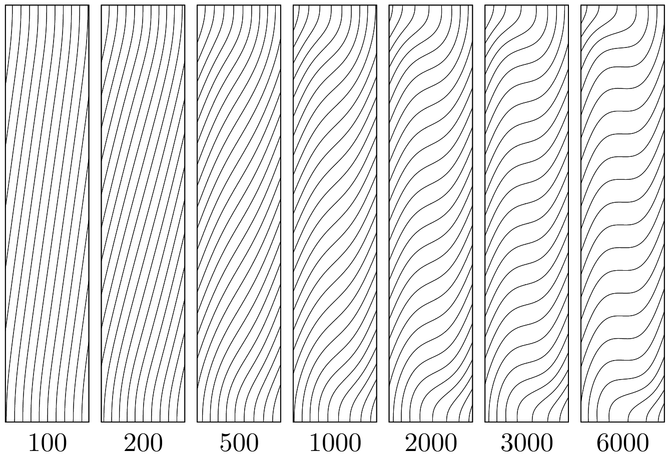

Fig. 4 Change in T with increasing Ra, as marked; $A = 5$, Pr = ∞.

We see that for small Ra, isotherms are nearly vertical and heat transfer is dominated by horizontal conduction; as Ra increases, the ensuing clockwise circulation tilts the isotherms and sets up a stable stratification in the core. Nevertheless, across this transition from conduction to convection, the system remains one-dimensional over most of the height, as illustrated in Fig. 5.

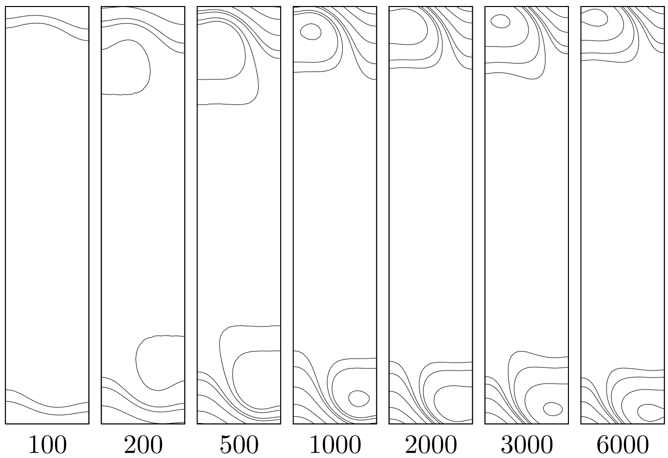

Fig. 5 Discrepancy from the fully developed temperature; parameters as in Fig. 4. Contours at $\pm 1, 2, 5, 10, 15 \times 10^{-2}$.

This is very different from the isothermal hot-and-cold wall case, in which increasing Ra advects 'floor-effects' up into the rest of the cavity, destroying the one-dimensional regime [12]; in that case, the size of the floor regions like those illustrated in Fig. 5 only grow with Ra [11, Fig. 5]. Here, this is offset by the increase in stratification with increasing Ra, which is not incompatible with the vertical wall boundary conditions (4).

Nusselt number

Define the local Nusselt number as the ratio of the imposed heat flux to that corresponding to pure conduction at the prevailing temperature difference; i.e.

$$\mathrm{Nu}(y) \equiv \left\{ T\left(\frac{-1}{2}, y\right) - T\left(\frac{+1}{2}, y\right) \right\}^{-1}. \qquad (14)$$

Then, from (7), the fully developed Nusselt number is

$$\mathrm{Nu}_{A=\infty} = s\,\frac{\sinh 2s + \sin 2s}{\cosh 2s - \cos 2s}; \qquad (15)$$

the parameter s can be eliminated between (15) and (10) to give the Nu–Ra heat transfer relation. For extreme s, this simplifies to

$$\mathrm{Nu}_{A=\infty} \sim \begin{cases} 1 + O(\mathrm{Ra}^2), & \mathrm{Ra} \to 0; \\ \mathrm{Ra}^{2/9}/2^{14/9}, & \mathrm{Ra} \to \infty, \end{cases} \tag{16}$$

which we take to define the conduction and convection heat transfer regimes for a tall cavity. The Ra $\to \infty$ limit here is equivalent to the KB result (but without the apparent dependence on A arising from a poor initial choice of length scale). The analytic and numerical heat transfer results are compared in Fig. 6.

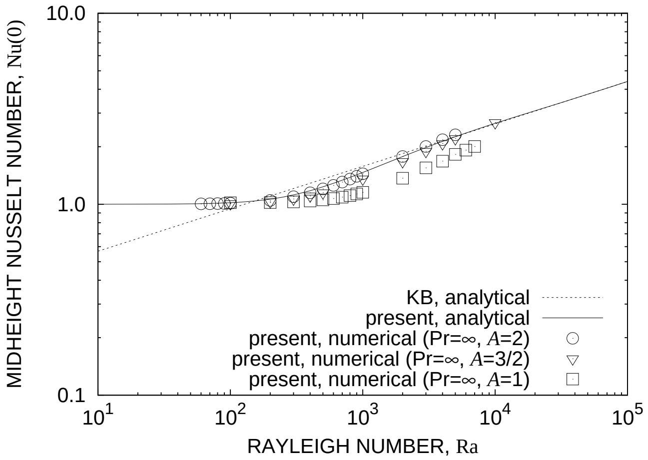

Fig. 6 Heat transfer relation for the evenly heated cavity.

Conclusions

By combining Lietzke's base solution with Kimura & Bejan's energy balance, a solution valid across the entire range of laminar Rayleigh numbers has been obtained. It is independent of the Prandtl number, and should apply for any cavity with tall enough aspect ratio. Numerical solutions have been presented to demonstrate the existence of such tall cavities.

For tall cavities, the conduction–convection transition occurs around Ra $\approx 10^2$–10^4; at higher Rayleigh numbers, the KB Nusselt formula is entirely adequate.

As far as the midheight Nusselt number is concerned, even $A = 2$ is approaching tall, and the shortest tall A appears to decrease with increasing Rayleigh numbers.

The new formula, resulting from the combination of Lietzke's base solution with Kimura & Bejan's energy balance, describes the variation of the Nusselt number very well for $A = 2$ and Pr $= \infty$, and would be expected to perform even better at any higher A.

Further work is required to determine how the minimum A for which the theory applies depends on the Rayleigh and Prandtl numbers. This could take the form of more numerical solutions, or, following Daniels [12] for the isothermal hot-and-cold wall case, a study of the eigenstructure of the governing equations linearized about the fully developed state; already though it is clear that whereas there the minimum A increases linearly with Ra, the behaviour here is complicated by the dependence of the base solution on Ra.

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